Econ 802
Lecture Notes on Chapter 2

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In This chapter we start The analysis of optimization for a competitive firm.

The key idea is The <u>profit function</u>, which gives The maximum possible profit as a function of prices:

$$\pi(p) \equiv \max_{y \in Y} \; p \cdot y = p \cdot y(p)$$ where $y(p)$ is a solution to The max problem.

Note: $p = (p_1 \cdots p_n) > 0$ is The price vector
$y = (y_1 \cdots y_n)$ is a production plan

If The solution $y(p)$ is unique we can study how The firm's supply/demand behavior is determined by prices

Some variations on The same general idea:

(1) <u>Short run profit function</u>:

$$\pi(p, z) \equiv \max_{y \in Y(z)} \; p \cdot y = p \cdot y(p, z)$$

where z is a vector of fixed inputs

(2) <u>Single-output profit function</u>:

$$\pi(p, w) \equiv \max_{x \geq 0} \; p f(x) - w \cdot x$$

where $p > 0$ is a scalar output price, $w = (w_1 \cdots w_n) > 0$ is a vector of input prices and $x = (x_1 \cdots x_n) \geq 0$ is a vector of input quantities.

(3) <u>Cost function:</u>

$$c(w,y) \equiv \min_{x \in V(y)} w \cdot x$$

where $y \geq 0$ is a scalar output

(4) <u>Short run cost function:</u>

$$c(w,y,z) \equiv \min_{(y,-x) \in Y(z)} w \cdot x$$

where $Y(z)$ is the short run production possibilities set.

Assuming the relevent functions are well-defined:

in case (1) $y(p,z)$ gives short run optimal behavior

in case (2), $x(p,w)$ shows how the optimal inputs vary with output price and input prices (These are called <u>unconditional</u> input demands because they do not depend on an output level)

To get the output supply function in case (2), we write

$$y(p,w) = f[x(p,w)]$$

[substitute $x(p,w)$ into the production function)

In case (3), $x(w,y)$ shows how the optimal inputs vary with input prices and output quantity (These are called <u>conditional</u> input demands because they depend on the output level)

In case (4), we write $x(w,y,z)$ and call these short run conditional input demand functions.

<u>Calculus techniques</u> (I strongly recommend that you read Ch. 27 of Varian on optimization)

Let's start with the simplest case: profit max in the long run for a single output.

Notation: I like to write the vector of partial derivatives for the production function at a point x^* as

$$\left[\frac{\partial f(x^*)}{\partial x_1} \cdots \frac{\partial f(x^*)}{\partial x_n}\right] \equiv \frac{\partial f(x^*)}{\partial x}$$

Varian calls the same thing $Df(x^*)$.

I like to write the Hessian matrix for the production function at a point x^* as

$$\left\| \frac{\partial^2 f(x^*)}{\partial x_i \partial x_j} \right\| \equiv \frac{\partial^2 f(x^*)}{\partial x^2}$$

Varian calls the same thing $D^2 f(x^*)$

Now consider $\max\limits_{x \geq 0} \; pf(x) - w \cdot x$

Let's ignore the non-negativity constraints $x \geq 0$ for the moment (I'll come back to this issue later).

The first order conditions are

FOC: $p \dfrac{\partial f(x^*)}{\partial x_i} = w_i$ for $i = 1 \ldots n$ or $p \dfrac{\partial f(x^*)}{\partial x} = w$.

If possible we would like to solve for x^* in terms of p and w. But first, consider second order conditions.

SOC: The Hessian matrix of the function being maximized must be negative semidefinite. This reduces to the Hessian of the production function.

So a <u>necessary</u> condition for x^* to be a solution is

$$h \frac{\partial^2 f(x^*)}{\partial x^2} h \leq 0 \quad \text{for any vector } h = (h_1 \ldots h_n)$$

To be explicit, let's write out the Hessian matrix:

$$\frac{\partial^2 f(x^*)}{\partial x^2} = \left\| \frac{\partial^2 f(x^*)}{\partial x_i \partial x_j} \right\| = \begin{bmatrix} \frac{\partial^2 f(x^*)}{\partial x_1^2} & \cdots & \frac{\partial^2 f(x^*)}{\partial x_1 \partial x_n} \\ \vdots & & \\ \frac{\partial^2 f(x^*)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(x^*)}{\partial x_n^2} \end{bmatrix}$$

To interpret the necessary SOC think of h as some small (local) deviation from x^*. The condition says that you cannot increase profit no matter which h you choose.

However, neg semi-definiteness is only a <u>necessary</u> condition for x^* to be a solution.

A <u>sufficient</u> condition is that $\frac{\partial^2 f(x^*)}{\partial x^2}$ be neg <u>definite</u>. <u>Equivalently</u>:

$$h \frac{\partial^2 f(x^*)}{\partial x^2} h < 0 \quad \text{for all } h \neq 0.$$

Note: I tend to be sloppy about my matrix notation and omit transpose signs. You should be able to figure out what I'm doing. I will try to be explicit about this when it is important.

Try to be careful about the distinction between necessary and sufficient conditions. If x^* is a solution it <u>must</u> satisfy the necessary condition. However, not everything that satisfies the nec condition must be a solution (there could be non-solutions for which it also holds).

A sufficient condition means exactly what it says. If x^* satisfies the sufficient condition then it is a solution. However there could be solutions that don't satisfy the sufficient condition (it is not necessary)

All of the calculus conditions discussed above are local in nature (h is a small deviation from x^*).

What about global conditions? The following things are true.

If f is concave then the necessary SOC holds at every point x and any x^* that satisfies the FOC is a global max (it solves the problem).

If f is strictly concave then the same things are true, and there is a unique solution x^*.

If f satisfies the sufficient SOC at all points (the Hessian is negative definite everywhere) then the same things are true, and x^* is a differentiable function of (p, w).

So going from the necessary SOC holding everywhere to concavity buys you the fact that anything satisfying FOC is a solution;
making the concavity strict buys you uniqueness;
and adding neg definiteness globally buys you the differentiability of the solution (using the implicit function theorem)

For more on all of this see the notes on my web site about concavity and optimization.

Note: if all we know about the production function is that it is quasi-concave (has convex input requirement sets) this tells us nothing about the necessary or sufficient SOC for profit max.

Intuition: The production function describes the upper boundary of the production possibility set Y from chapter 1. The nec/suff SOC and the concavity or strict concavity of the production function give us information about the curvature of this upper boundary. Quasi-concavity does not provide such information.

Hazards

The next thing I want to discuss is a series of problems that can arise with the methods described above.

(1) Non-differentiable production functions

Obviously if you cannot take derivatives of the production function, you can't use calculus. The most common example of this kind is the Leontief production function:

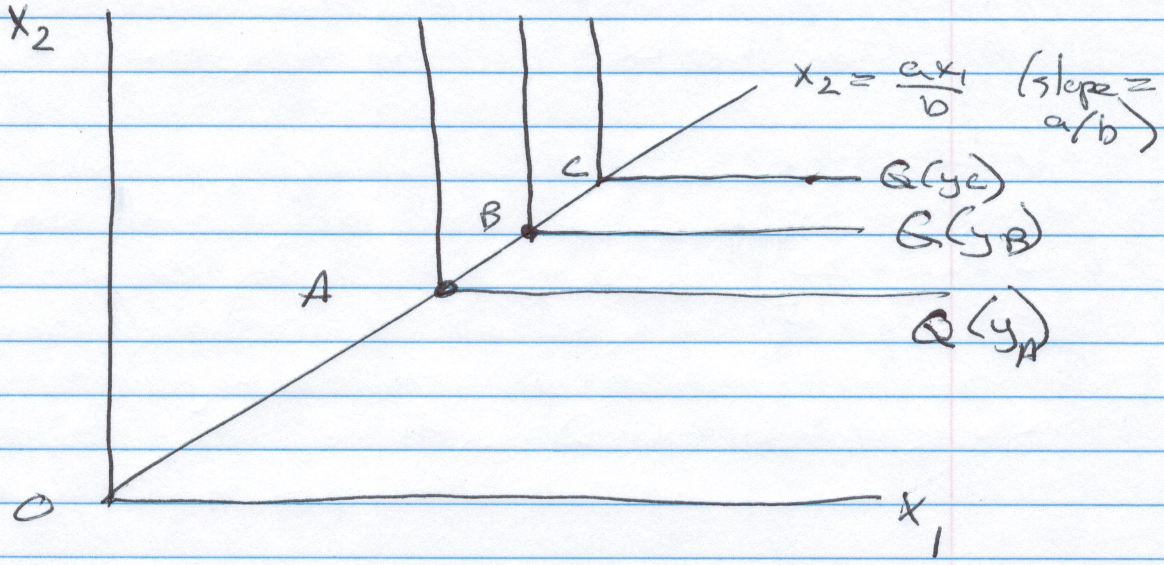
$$y = \min\{ax_1, bx_2\}$$
$$\text{where } a > 0, \ b > 0.$$

I want to spend some time on the isoquants and marginal products for this function so you will see how it works.

Start with any point $x = (x_1, x_2) > 0$ such that $ax_1 = bx_2$. In this case, the output must be

$$y = ax_1 = bx_2.$$

Furthermore, the point must be on the ray from the origin described by $x_2 = \frac{ax_1}{b}$:

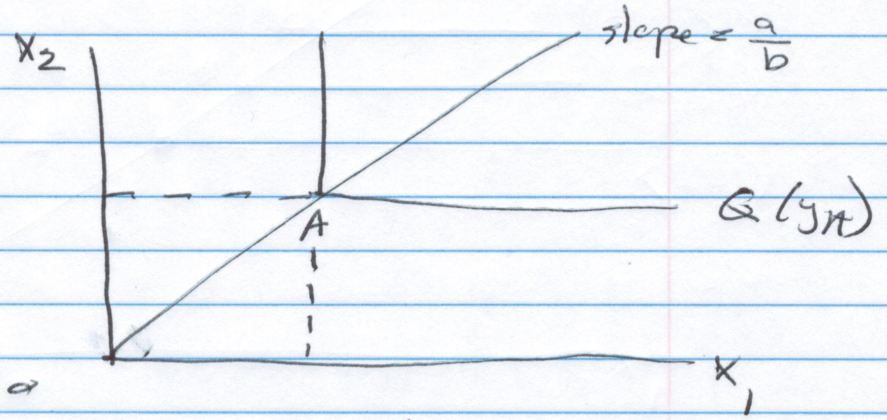


Call the initial point A. Now increase x_1 while keeping x_2 constant. This gives $y = bx_2 < ax_1$. Output does not increase because it is determined by the smaller of ax_1 and bx_2. So all points with $y = bx_2 < ax_1$ are on the same isoquant as point A. Similarly, if we start from A and increase x_2 while keeping x_1 constant, we get $y = ax_1 < bx_2$. So again these points are on the same isoquant as A.

It should be clear that if you start from point A and increase <u>both</u> inputs simultaneously, you will get more output, so you move to a higher isoquant, as shown in the graph above for points B, C, etc.

Now let's think about marginal products. Again start from point A. If you are anywhere along the horizontal part of the isoquant $Q(y_A)$ and you increase x_1, output does not change, so the marginal product of x_1 is zero. This gives a well-defined partial derivative $\frac{\partial f(x)}{\partial x_1} = 0$. ($MP_1$ is zero)

What may be less obvious is MP_1 to the left of point A:



Think about the dashed horizontal line above. Here we have $y = ax_1 < bx_2$ so output is determined by x_1. Here again we have a well-defined partial derivative $\frac{\partial f(x)}{\partial x_1} = a > 0$. ($MP_1$ is positive)

The problem arises at point A, where the partial derivative $\frac{\partial f(x)}{\partial x_1}$ is discontinuous; it jumps from $a > 0$ down to zero.

A similar problem arises for $\frac{\partial f(x)}{\partial x_2}$. It is $b > 0$ on the vertical dashed line, discontinuous at A, and zero along the vertical part of $Q(y_A)$.

Unfortunately, we will see later that a profit maximizing firm wants to operate along the ray $x_2 = \frac{ax_1}{b}$ where the production function is non-differentiable, so calculus is not useful in this case.

2. Interior and boundary solutions

I said earlier that we were ignoring the non-negativity constraints $x \geq 0$ (see p. 3)

This is OK if we look at the FOC and get a solution where $x^* \geq 0$. In this situation we are not violating any constraints. But if we get $x_i^* < 0$ for one or more inputs, we have a problem and we have to impose non-negativity explicitly.

Notice that $x_i \geq 0$ is not an equality constraint, so Lagrange multipliers are inappropriate.

However, we can use a similar idea:

Kuhn-Tucker conditions.

Note: I will only discuss FOC here. These are necessary not sufficient. However if we combine the K-T FOCs with an assumption of concavity for the production function, the K-T FOCs will be sufficient.

Consider the same problem as before:

$$\max_{x \geq 0} \; pf(x) - w \cdot x$$

Now add a vector of Kuhn-Tucker multipliers $\mu = (\mu_1 \ldots \mu_n)$ and multiply by the vector x:

$$pf(x) - w \cdot x + \mu x$$

The FOC become

$$p\frac{\partial f(x^*)}{\partial x_i} - w_i + \mu_i = 0, \quad i = 1..n$$

$$\underline{\text{and}} \quad \mu_i \geq 0, \; x_i \geq 0, \; \mu_i x_i = 0, \; \text{all } i = 1..n.$$

The restrictions in the second line are part of the FOCs.

What does this imply?

① If $x_i^* > 0$ then we must have $\mu_i = 0$, so we go back to the usual FOC without K-T.

② If $\mu_i > 0$ then we must have $x_i^* = 0$. This is called a boundary solution for input i.

In case ②, we set

$$p\frac{\partial f(x^*)}{\partial x_i} - w_i < 0.$$

This implies that when $\mu_i > 0$ and $x_i^* = 0$ the firm would have <u>less</u> profit if it went to $x_i > 0$. Since this is bad, the firm remains at $x_i^* = 0$.

If all we know is $x_i^* = 0$, we could have either $\mu_i = 0$ or $\mu_i > 0$ (both are consistent with $\mu_i x_i^* = 0$). So then all we know is

$$p\frac{\partial f(x^*)}{\partial x_i} - w_i \leq 0$$

where x_i is "small"

Thus if the firm goes to $x_i > 0$, profit might stay the same or it might fall, but it will not increase.

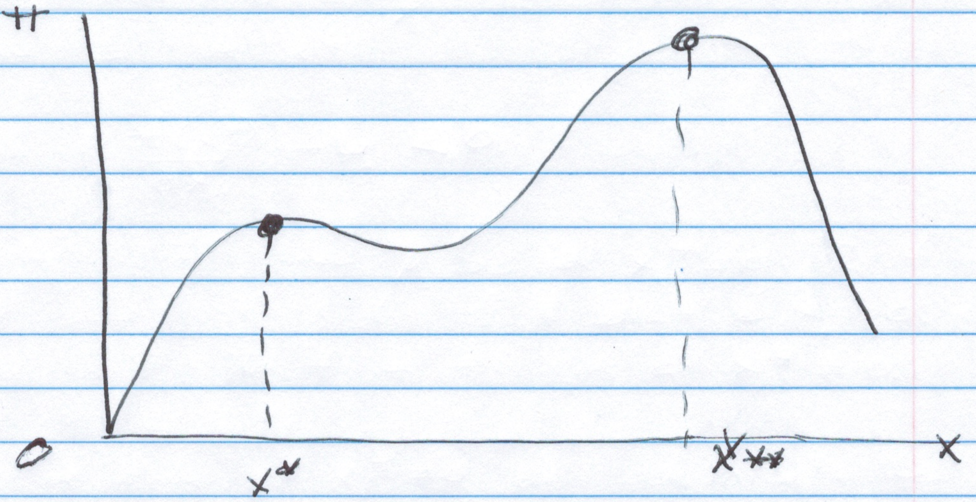
In practice there is no mechanical way to solve such problems. You have to check each possible case $\mu_i > 0$ or $\mu_i = 0$ for all i and see whether there is an x^* that satisfies all of the FOCs.

3 Non-uniqueness of optimal points

Unless you have concavity of the production function or some other special restriction you need to be alert to the possibility of multiple solutions. This could take a couple of forms.

(a) local versus global solutions

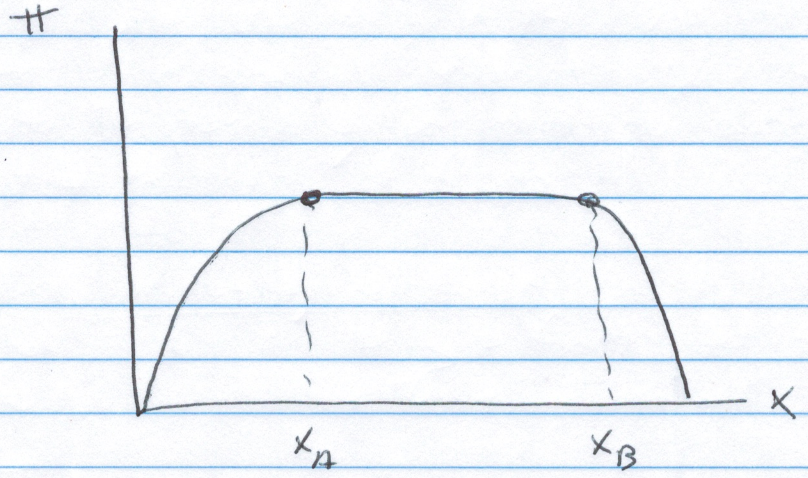
Think of profit as being a function of some scalar x. Suppose it looks like this:



Clearly x^{**} is the unique global solution. However x^* will satisfy both the FOC and the sufficient SOC even though it does not maximize π. The problem is that calculus only gives you local information about the shape of the objective function near a candidate point x^*. It does not provide global information that would allow you to distinguish between x^* and x^{**}. In a case like this, you need to calculate the value of the objective function at both x^* and x^{**} to see which gives a higher value.

(b) An interval of solutions

Another thing that can happen is something like this:



Any point with $x_A \le x \le x_B$ gives the maximum possible profit, so we have many solutions (a continuum). In a situation like this the firm's choice of x is non-unique and we would have problems describing how the firm's behavior changes in response to prices.

Note that concavity of the production function does not rule out situations like this. It does ensure that if x^* satisfies the FOC it really is a solution. But we would need strict concavity to get uniqueness (or at least to guarantee it).

Normally the way you would know that there could be multiple solutions is that more than one point satisfies the FOC. This is a clue that you may have complications like more than one point that maximizes the objective or some points that give a max while others give a min, etc.

4) A solution may not exist.

Consider the problem $\max_{y \in Y} p \cdot y$.

How do we know this problem ever has a solution? If it does not, then the profit function $\pi(p)$ is undefined, and our whole theory of firm behavior falls apart.

There is one simple case where this problem occurs. Suppose (a) we have constant returns to scale and (b) positive profit is possible.

In particular, suppose $pf(x^0) - wx^0 > 0$ for some input vector $x^0 = (x_1^0 \ldots x_n^0)$. Then CRS implies that for any $t > 0$ we have

$$pf(tx^0) - w \cdot (tx^0) = tpf(x^0) - tw \cdot x_0$$
$$= t\{pf(x^0) - wx^0\} > 0.$$

Clearly there is no finite solution here; we can always make profit higher by making t higher.

This problem does not arise if we have CRS and the maximum possible profit is zero. Then scaling $t > 0$ up or down still gives zero. BUT the scale of the firm is indeterminate (there are multiple solutions)

Also note that we could have CRS in the long run, but DRS in the short run due to some fixed inputs. In this case the SR profit function $\pi(p, z)$ could still be well-defined, even if the LR function $\pi(p)$ is not.

Properties of Input Demand and Output Supply

A key set of questions about the theory of the firm involves comparative statics: when the (exogenous) prices change, how do the (endogenous) quantities respond?

There are 3 general ways to approach such questions:
① Manipulation of first order conditions
② The duality method
③ The algebraic method
I will start with ① and then go to ③. Method ② will be addressed in chapter 3.

The FOC method is the hardest and requires the most assumptions. However, it is often used and any economist needs to understand it.

Once again, start with a competitive firm having a single output, and assume it solves

$$\max_x \; pf(x) - w \cdot x$$

I will assume optimal solutions are interior, so we don't need Kuhn-Tucker multipliers. I will also assume differentiability.

FOC: $\underset{1\times 1}{p}\,\underset{1\times n}{\dfrac{\partial f(x)}{\partial x}} = \underset{1\times n}{w}$

[I will be explicit about the dimensionality of vectors and matrices for clarity.]

Now suppose we can use the implicit function theorem to solve for the unconditional input demands $x(p,w)$. I will say more below about when this is valid. In what follows I suppress the output price p because it will not be changing, so I will just write $x(w)$.

By definition, $p\frac{\partial f[x(w)]}{\partial x} \equiv w$ (this is an identity)

Now differentiate both sides with respect to the vector of input prices w:

$$p \frac{\partial^2 f[x(w)]}{\partial x^2} \frac{\partial x(w)}{\partial w} = I$$

1×1 $\quad n\times n$ $\quad n\times n$ $\quad n\times n$

where $\frac{\partial x}{\partial w} = \begin{bmatrix} \frac{\partial x_1}{\partial w_1} & \cdots & \frac{\partial x_1}{\partial w_n} \\ \vdots & & \\ \frac{\partial x_n}{\partial w_1} & \cdots & \frac{\partial x_n}{\partial w_n} \end{bmatrix}$ (This is called the substitution matrix)

Next suppose the Hessian matrix $\frac{\partial^2 f}{\partial x^2}$ is non-singular. Technically we need this assumption in order to use the implicit function theorem, so really we already assumed it. In this case we have

$$\frac{\partial x(w)}{\partial w} = \left[p \frac{\partial^2 f[x(w)]}{\partial x^2}\right]^{-1}$$

A typical element of this matrix $\frac{\partial x_i}{\partial w_j}$ gives the rate at which the firm's demand for input i changes in response to a change in the price of input j (This could be positive or negative)

Conclusions from this result:

(a) <u>symmetry</u>. Because the Hession $\frac{\partial^2 f}{\partial x^2}$ is symmetric, so is its inverse, and therefore so is $\frac{\partial x}{\partial w}$. Hence $\frac{\partial x_i}{\partial w_j} = \frac{\partial x_j}{\partial w_i}$ for all i, j.

[Note that this is a non-obvious implication of profit maximization.]

(b) <u>Negative definiteness</u>. If $x(w)$ is actually optimal, it must satisfy the necessary SOC, which is that $\frac{\partial^2 f}{\partial x^2}$ is negative semi-definite.

But we had to assume $\frac{\partial^2 f}{\partial x^2}$ was non-singular in order to invert it, and if a matrix is both neg semi-def <u>and</u> non-singular, then it is <u>negative definite</u>. Furthermore, the inverse of a neg def matrix is also neg def. Therefore, $\frac{\partial x}{\partial w}$ is negative definite.

(c) <u>Downward sloping input demand curves</u>

Because $\frac{\partial x}{\partial w}$ is neg def, its diagonal elements must be strictly negative. Therefore $\frac{\partial x_i}{\partial w_i} < 0$ for all $i = 1..n$. This means that if the price of input i rises, the quantity of input i falls - so the firm has a downward sloping unconditional demand curve for each of its inputs.

Note: if we wanted to know the effects of input prices w on output y, we could write

$$\frac{\partial y(p,w)}{\partial w} = \frac{\partial}{\partial w} f[x(p,w)] = \frac{\partial f[x(p,w)]}{\partial x} \cdot \frac{\partial x(p,w)}{\partial w}$$

(last line follows from FOC) $= \frac{w}{p} \frac{\partial x(p,w)}{\partial w}$

The Algebraic Method

This approach is easy and does not require calculus.
Suppose we have a finite data set
(p^t, y^t) for $t = 1 \ldots T$
where p^t is the price vector at time t
y^t is the production plan at time t

[We are now back to the notation

$$\max_{y \in Y} p \cdot y \quad \text{where } p = (p_1 \ldots p_n) \text{ and } y = (y_1 \ldots y_n)]$$

Assume we don't know the true production possibility set Y. But obviously, anything the firm does must be feasible.
Does profit maximization imply anything that should be true in the data?
Does it have observable implications?
Could this assumption ever be refuted by observations?

Some things must be true.
If y^t maximizes profit at prices p^t, then no other feasible production plan can give more profit.
In particular, we should have

$$p^t y^t \geq p^t y^s \quad \text{for all } s = 1 \ldots T \text{ and } t = 1 \ldots T$$

This condition is called the Weak Axiom of Profit Maximization (WAPM)

If this was ever violated in the data for some (s, t), then the firm would not be maximizing profit in period t (there would be some feasible plan y^s that would yield higher profit).

Suppose WAPM holds in the data, and choose some particular t and s. This gives

$$\left.\begin{array}{l} p^t y^t \geq p^t y^s \\ \text{and} \quad p^s y^s \geq p^s y^t \end{array}\right\} \Rightarrow \begin{array}{l} p^t(y^t - y^s) \geq 0 \\ -p^s(y^t - y^s) \geq 0 \end{array}$$

Sum the second pair of inequalities to obtain

$$(p^t - p^s)(y^t - y^s) \geq 0$$

or $\underset{1\times n}{\Delta p}\,\underset{n\times 1}{\Delta y} \underset{1\times 1}{\geq 0}$ where $\Delta p \equiv p^t - p^s$, $\Delta y \equiv y^t - y^s$

For instance: if $\Delta p_j > 0$ and $\Delta p_i = 0$ for all $i \neq j$ then we must have $\Delta y_j \geq 0$.

This says that output supply curves cannot slope down and input supply ~~curves cannot~~ slope up.

[if $y_j < 0$ so j is an input then $\Delta p_j > 0 \Rightarrow \Delta y_j \geq 0$ so in an algebraic sense y_j increases, meaning that it gets closer to zero. So the absolute value of the input quantity decreases, and the firm uses less of input j]

Cool things about this method:

1. It is a global result. We didn't use calculus so we are not limited to small price changes
2. We don't care about differentiability
3. We are not limited to one output
4. We don't need to know anything about the true Y set (monotonicity, convexity, etc)

It can be shown that WAPM is the *only* observable implication of profit maximization, in the following sense:

(a) if it is violated, we *cannot* have profit max

(b) if it is satisfied in the data there exists *some* technology Y for which the data are consistent with profit max.

I want to show why (b) is true. Varian calls this "recoverability" and discusses it in section 2.6 (pp. 36-38).

Suppose we have data $\{p^t, y^t\}$ for $t = 1..T$ and the data satisfy WAPM so $p^t y^t \geq p^t y^s$ for all t, s.

Is there *some* production set Y that the firm *could* have that would be consistent with its observed choices?

If so, can we construct Y so that it is closed, convex, and monotonic?

Yes. In fact there will generally be *more than one* Y that could be the true production possibility set.

Let's take a simple case with one input (y_1) one output (y_2), and two time periods ($t = 1, 2$)

Write the data as $\{p^1, y^1; p^2, y^2\}$ where $p^1 > 0$ and $p^2 > 0$.

In period $t = 1$, the firm has profit $\pi^1 = p^1 y^1$

In period $t = 2$, the firm has profit $\pi^2 = p^2 y^2$.

Now define the isoprofit line for period 1 to be the set of all production plans y such that $p^1 y = \pi^1$. Writing this out gives

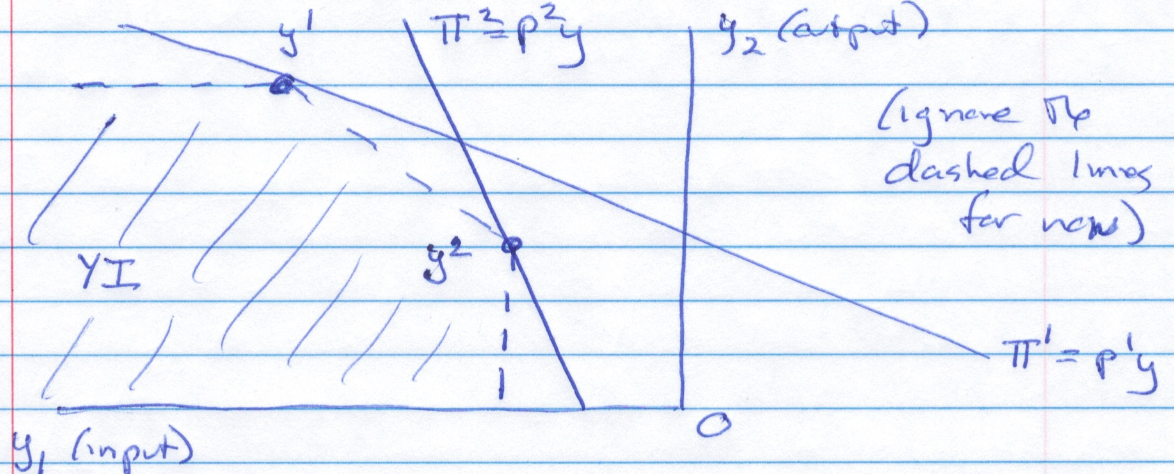
$$p_1^1 y_1 + p_2^1 y_2 = \pi^1 \quad \text{or} \quad y_2 = \frac{\pi^1}{p_2^1} - \frac{p_1^1 y_1}{p_2^1}$$

so the slope is negative and determined by the price ratio in period 1: $\frac{p_1^1}{p_2^1}$

Likewise define the isoprofit line for period 2 to be the set of all y such that $p^2 y = \pi^2$ or

$$y_2 = \frac{\pi^2}{p_2^2} - \frac{p_1^2 y_1}{p_2^2}$$

Let's draw a graph showing these isoprofit lines:



I am arbitrarily assuming the price ratio gives a steeper line in period 2, but this is not important. What matters is that the isoprofit line in period 1 must pass through y^1 and the isoprofit line in period 2 must pass through y^2. Furthermore, WAPM implies that y^2 must be on or below the line for period 1 because if y^2 was above the isoprofit line for period 1, it would be more profitable than y^1, which is a contradiction.

Similarly WAPM implies that y^1 must be on or below the isoprofit line for period 2. If y^1 was above this line at the period 2 prices p^2 it would give more profit than π^2, which would contradict WAPM.

Algebraically we are saying

$$\pi^1 = p^1 y^1 \geq p^1 y^2 \quad \text{and}$$
$$\pi^2 = p^2 y^2 \geq p^2 y^1 .$$

Now ask: what is the smallest closed, convex, and monotone set that contains y^1 and y^2? Call this set YI. The true Y must be at least this big.

Go back to the graph and look at the dashed lines.
If we require YI to be convex, it must contain all points on the dashed line segment between y^1 and y^2.
If we require YI to be monotonic, it must contain the vertical dashed line below y^2 and the horizontal dashed line to the left of y^1, as well as all points to the southwest of these dashed lines (with $y_1 \leq 0$ and $y_2 \geq 0$).
If we require YI to be closed, then it must contain its boundary.
So the smallest possible Y set with these properties is YI as shown in the graph.

Is YI consistent with the firm's observed behavior? Yes. At prices p^1 the firm cannot reach a higher profit than π^1, and at prices p^2 it cannot do better than π^2.

What is the <u>largest</u> closed, convex, and monotonic set that could be consistent with the firm's observed behavior? Call this set YO.

① YO cannot contain anything above the isoprofit line for period 1 (if it did, y^1 would not be profit max at prices p^1)

② YO cannot contain anything above the isoprofit line for period 2 (if it did, y^2 would not be profit max at prices p^2)

So YO is the set of all points on or below <u>both</u> of the isoprofit lines. It is easy to check that this set is closed, convex, and monotonic.

Is YO consistent with the firm's observed behavior?
Yes. y^1 maxes profit at p^1
and y^2 maxes profit at p^2
(although in both cases, the solution is non-unique)

We interpret YI and YO as the inner and outer bounds on the true set Y (which we do not observe directly).

In general, the more data we have the closer YI and YO will be to each other, and the more precisely we can approximate the true Y.

The reasoning described above extends to many inputs and outputs, many time periods etc.

A Cobb-Douglas example

I want to end this discussion of Chapter 2 by exploring a particular technology.

The general Cobb-Douglas production function is written as

$$y = f(x) = x_1^{\alpha} x_2^{\beta} \qquad \text{where } \alpha > 0,\ \beta > 0$$

(note that you can also multiply the right hand side by some positive constant, but this won't affect the points I want to make, so I simplify the notation by omitting it.)

Suppose the firm maximizes $pf(x) - wx$ where $p > 0$ is a scalar, $x = (x_1, x_2) \geq 0$ is the input vector and $w = (w_1, w_2) > 0$ is the input price vector.

If we jump right in and take derivatives, the FOC are

$$p f_1(x) - w_1 = 0$$
$$p f_2(x) - w_2 = 0$$

or $p\alpha x_1^{\alpha-1} x_2^{\beta} - w_1 = 0$

and $p\beta x_1^{\alpha} x_2^{\beta-1} - w_2 = 0$

Solving for (x_1, x_2) gives

$$x_1^* = \left[\frac{1}{p}\left(\frac{w_1}{\alpha}\right)^{1-\beta}\left(\frac{w_2}{\beta}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta-1}}$$

$$x_2^* = \left[\frac{1}{p}\left(\frac{w_1}{\alpha}\right)^{\alpha}\left(\frac{w_2}{\beta}\right)^{1-\alpha}\right]^{\frac{1}{\alpha+\beta-1}}$$

make sure you can do the algebra here!

Some things to notice about this solution:

1. It is possible to obtain a <u>unique</u> solution from the FOC <u>if</u> $\alpha+\beta-1 \neq 0$. But if this is zero, we have undefined exponents, which suggests there might be a problem.
2. Clearly $x_1^* > 0$ and $x_2^* > 0$ when $\alpha+\beta-1 \neq 0$, so it was OK to ignore the non-negativity constraints $x_1 \geq 0$ and $x_2 \geq 0$ (we didn't need Kuhn-Tucker multipliers)
3. It was OK to take derivatives because the production function is differentiable.
4. The solution has the form
$$x(p,w) = [x_1(p,w),\ x_2(p,w)]$$
So we might think we have found the unconditional input demand functions.
5. <u>BUT</u> we have not yet checked the SOCs. We don't know whether we have a max, a min, or what.
6. Furthermore, there is something strange going on when $\alpha+\beta > 1$. In this case, an increase in w_1 gives an increase in x_1. This contradicts the comparative static results we obtained earlier, which say input demands can't slope up. Likewise, when $\alpha+\beta > 1$ an increase in w_2 gives an increase in x_2. This is a sign that something is wrong.

To investigate further, look at the SOC. The <u>necessary</u> SOC says $\frac{\partial^2 f(x^*)}{\partial x^2}$ must be neg <u>semidef</u>; the <u>sufficient</u> SOC says at x^* is neg. <u>def.</u>

To see whether these hold, we need the second derivatives f_{11}, f_{12}, f_{21}, f_{22}

The Hessian is neg semi-def if
$$f_{11} \leq 0,\ f_{22} \leq 0,\ \text{and } f_{11}f_{22} - f_{12}^2 \geq 0.$$

The Hessian is neg def if
$$f_{11} < 0,\ f_{22} < 0,\ f_{11}f_{22} - f_{12}f_{21} > 0$$
These conditions imply $\alpha < 1$, $\beta < 1$, $1 - \alpha - \beta > 0$.

Look further at this last set of results. If $\alpha + \beta < 1$, the exponent $\frac{1}{\alpha+\beta-1}$ is negative in the solution x_1^*, x_2^*. Now these results make sense: when $w_1 \uparrow$ we have $x_1 \downarrow$ and when $w_2 \uparrow$ we have $x_2 \downarrow$
So the comparative statics are going in the right direction.

You should be able to show that $\alpha + \beta < 1$ implies decreasing returns to scale. So when the Cobb-Douglas function has DRS, everything works fine: the sufficient SOC holds, we really are maximizing profit, and the comparative statics make sense.

The problem when $\alpha + \beta > 1$ is that we have increasing returns (prove this!). With IRS the profit max problem has no solution, so the FOC are giving nonsense. Not only are we violating the sufficient SOC, we are also violating the necessary SOC.

The last mystery involves the case $\alpha + \beta = 1$ where the exponents in x_1^* and x_2^* are undefined. In this case we have constant returns (prove this!). Although the necessary SOC is satisfied, the <u>sufficient</u> SOC is not.

We know from earlier results (see p. 13) that with CRS, a solution to the profit max problem may not exist.

However, if the maximum possible profit is zero, a solution <u>does</u> exist. Even in this case though, the solution (x_1^*, x_2^*) makes no sense due to the issue with the exponents. The problem is that the solution is <u>not</u> unique. If we scale the inputs and output up and down by some scalar $t > 0$, we still get zero profit, so we still have a max. Thus the appearance of a unique solution (x_1^*, x_2^*) is misleading.

The moral of the story is that even in a simple case like the Cobb-Douglas production function, it does not always make sense just to take derivatives and solve the first order conditions. You need to think carefully about what you are doing, why you are doing it, and whether you are getting results that make economic sense.

That's all for Chapter 2!